

The Binomial Distribution

Definition: A *binomial experiment* is a probability experiment for which the following four properties hold:

- (1) There are fixed number of trials.
- (2) Each trial has only two outcomes. We usually denote these outcomes as “success” and “failure.”
- (3) The trials must be independent.
- (4) The probability of success must remain constant for each trial.

The associated probability distribution is called a *binomial distribution*.

Note: While we call the outcomes “success” and “failure,” you should be careful not to assume that successes are always good things (or failures always bad). For example, if we are studying liver tumors, and we find a tumor in the liver of a rat, we call it a success. I suspect that the rat, however, would disagree.

Example 1: Examples of binomial experiments:

- Flip a coin. (H or T)
- Birth of a baby. (M or F)
- Sporting events. (W or L)
- Medical trial. (S or F)

Let X be the number of successes in n trials of a probability experiment. Define p to be the probability of a success (and $q = 1 - p$ is therefore the probability of failure). Then the probability of X successes is:

$$P(X) = \frac{n!}{(n-X)!X!} \cdot p^X \cdot q^{n-X}$$

[What the heck is “ $n!$ ”? This represents the *factorial* of n . For any natural number n , $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$, and for 0 , $0! = 1$.]

Example 2: If a coin is flipped 4 times, what is the probability of getting exactly two heads?

Recalling Example 2 of our last set of notes, we see that 6 outcomes (out of 16) produced exactly two heads. So the answer is $\frac{6}{16} = .375$. But let's determine this using our formula (since on other problems, writing out the entire sample space will not be convenient).

We have $n = 4$, $X = 2$, $p = .50$, $q = .50$. So,

$$\begin{aligned}P(2) &= \frac{4!}{(4-2)!2!} \cdot (.5)^2 \cdot (.5)^{4-2} \\ &= \frac{24}{2 \cdot 2} \cdot (.25) \cdot (.25) \\ &= 6 \cdot (.25) \cdot (.25) = .375\end{aligned}$$

Example 3: On a multiple-choice test there are 10 questions and each question has 4 possible answers. (a) What is the probability of guessing and getting exactly 5 correct? (b) What is the probability of guessing and getting at least 8 correct?

$n = 10$, $X = 5$, $p = .25$, $q = .75$. So,

(a)

$$\begin{aligned}P(5) &= \frac{10!}{(10-5)!5!} \cdot (.25)^5 \cdot (.75)^{10-5} \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot (.00097) \cdot (.23730) \\ &= .058\end{aligned}$$

(b)

$$\begin{aligned}P(\geq 8) &= P(8) + P(9) + P(10) \\ &= \frac{10!}{2!8!} \cdot (.25)^8 \cdot (.75)^2 + \frac{10!}{1!9!} \cdot (.25)^9 \cdot (.75)^1 + \frac{10!}{0!10!} \cdot (.25)^{10} \cdot (.75)^0 \\ &= 45 \cdot (.0000153) \cdot (.5625) + 10 \cdot (.000003815) \cdot (.75) + 1 \cdot (.000000954) \cdot 1 \\ &= .0003873 + .00002861 + .000000954 \\ &= .000417 \approx .0004\end{aligned}$$

Example 4: It was determined that 60% of Americans who are victims of health care fraud are seniors. If 15 victims are randomly selected, find the probability that exactly 4 are seniors.

$n = 15$, $X = 4$, $p = .60$, $q = .40$. So from chart, $P(4) = .007$.

Standard Deviation of a Binomial Distribution

$\mu = np$, $\sigma^2 = npq$, and (of course) $\sigma = \sqrt{npq}$.

Example 5: Find the mean, variance, and standard deviation for our test-taking example (Example 3).

The mean would be $\mu = 10 \cdot (.25) = 2.5$, the variance would be $\sigma^2 = 10(.25)(.75) = 1.875$, and the standard deviation would be $\sigma = \sqrt{1.875} = 1.37$.